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THE TANGENT TO THE PARABOLA.

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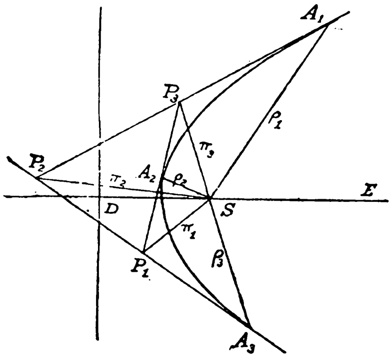
It is proposed in this paper to discuss, by the quaternion method, the relations between three intersecting tangents to the parabola.



Suppose tangents to be drawn at any three points as A_1, A_2, A_3; designate the vectors from the focus to these points by ρ_1, ρ_2, ρ_3 respectively. These tangents intersect each other at P_1, P_2, P_3; designate the vectors from the focus to these points of intersection by π_1, π_2, π_3 respectively.

Let α be a unit vector parallel to the axis, and β a unit vector parallel to the directrix; then the equation of the parabola referred to the focus as origin is

$$\rho = \frac{1}{4a} y^2 \alpha + y\beta - a\alpha = \frac{1}{4a} (y^2 - 4a^2)\, \alpha + y\beta \,, \tag{1}$$

in which ρ is any radius-vector and $a = \frac{1}{2} SD =$ the distance from the focus to the vertex. The vector along the tangent is $\frac{y}{2a} \alpha + \beta$.

Since ρ_1 and ρ_2 are vectors to the curve we have, by (1),

$$\rho_1 = \frac{1}{4a} (y_1^2 - 4a^2)\, \alpha + y_1\beta \,, \qquad \rho_2 = \frac{1}{4a} (y_2^2 - 4a^2)\, \alpha + y_2\beta \,;$$

from which equations, since $\rho^2 = -(\mathrm{T}\rho)^2$,

$$\mathrm{T}\rho_1\, \mathrm{T}\rho_2 = \frac{1}{16a^2} (y_1^2 + 4a^2)(y_2^2 + 4a^2) \,. \tag{2}$$

Again

$$\pi_1 = \rho_2 + A_2 P_1 = \frac{1}{4a} (y_2^2 - 4a^2)\, \alpha + y_2\beta + z \left(\frac{1}{2a} y_2\alpha + \beta\right),$$

$$\pi_1 = \rho_3 + A_3 P_1 = \frac{1}{4a} (y_3^2 - 4a^2)\, \alpha + y_3\beta + x \left(\frac{1}{2a} y_3\alpha + \beta\right),$$

in which x and z are unknown coefficients, easily determined by vector comparison, thus: $x = \frac{1}{2} (y_2 - y_3)$, $z = \frac{1}{2} (y_3 - y_2)$, and

$$\pi_1 = \frac{1}{4a} (y_2 y_3 - 4a^2)\, \alpha + \tfrac{1}{2} (y_2 + y_3)\, \beta \,, \tag{3}$$

whence, and by a cyclic change of suffices,

$$\left.\begin{aligned}(\mathrm{T}\pi_1)^2 &= \frac{1}{16a^2}(y_2^2+4a^2)(y_3^2+4a^2)\\ (\mathrm{T}\pi_2)^2 &= \frac{1}{16a^2}(y_3^2+4a^2)(y_1^2+4a^2)\\ (\mathrm{T}\pi_3)^2 &= \frac{1}{16a^2}(y_1^2+4a^2)(y_2^2+4a^2)\end{aligned}\right\} \quad (4)$$

Hence, by (2) and the analogous formulae,

$$(\mathrm{T}\pi_1)^2 = \mathrm{T}\rho_2\,\mathrm{T}\rho_3, \quad (\mathrm{T}\pi_2)^2 = \mathrm{T}\rho_3\,\mathrm{T}\rho_1, \quad (\mathrm{T}\pi_3)^2 = \mathrm{T}\rho_1\,\mathrm{T}\rho_2, \qquad (5)$$

or *the distance from the focus of a parabola to the intersection of two tangents is a mean proportional between the radii-vectores to the points of contact.*

Equation (3) shows that *the distance of the point of intersection of two tangents, from the axis, is the arithmetical mean between the ordinates to their points of contact.*

From (5) we deduce

$$\mathrm{T}\pi_2\,\mathrm{T}\pi_3 = \mathrm{T}\rho_1\,\mathrm{T}\pi_1, \quad \mathrm{T}\pi_3\,\mathrm{T}\pi_1 = \mathrm{T}\rho_2\,\mathrm{T}\pi_2, \quad \mathrm{T}\pi_1\,\mathrm{T}\pi_2 = \mathrm{T}\rho_3\,\mathrm{T}\pi_3. \qquad (6)$$

By (3) and the analogous formulae

$$P_2P_3 = \pi_3 - \pi_2 = \frac{1}{4a}y_1(y_2-y_3)\,\alpha + \tfrac{1}{2}(y_2-y_3)\,\beta,$$

$$A_3P_1 = \pi_1 - \rho_3 = \frac{1}{4a}y_3(y_2-y_3)\,\alpha + \tfrac{1}{2}(y_2-y_3)\,\beta.$$

Squaring and taking the tensors of these two expressions,

$$\overline{P_2P_3}^2 = \frac{1}{16a^2}(y_3-y_2)^2(y_1^2+4a^2), \ \overline{A_3P_1}^2 = \frac{1}{16a^2}(y_3-y_2)^2(y_3^2+4a^2). \quad (7)$$

Then, by (4) and (7),

$$\frac{(\mathrm{T}\pi_3)^2}{\overline{P_2P_3}^2} = \frac{y_2^2+4a^2}{(y_3-y_2)^2}, \ \frac{(\mathrm{T}\pi_1)^2}{\overline{A_3P_1}^2} = \frac{y_2^2+4a^2}{(y_3-y_2)^2}, \qquad (8)$$

and by (6) and (8),

$$\frac{P_2P_3}{A_3P_1} = \frac{\mathrm{T}\pi_3}{\mathrm{T}\pi_1} = \frac{\mathrm{T}\pi_2}{\mathrm{T}\rho_3}, \quad \frac{P_3P_1}{A_1P_2} = \frac{\mathrm{T}\pi_1}{\mathrm{T}\pi_2} = \frac{\mathrm{T}\pi_3}{\mathrm{T}\rho_1}, \quad \frac{P_1P_2}{A_2P_3} = \frac{\mathrm{T}\pi_2}{\mathrm{T}\pi_3} = \frac{\mathrm{T}\pi_1}{\mathrm{T}\rho_2}. \qquad (9)$$

From these equations the following relations between angles readily appear:

$$\begin{aligned}A_1P_2X^* &= P_2SA_3 = A_1SP_2 = P_3SP_1,\\ SA_1P_2 &= SP_2A_3 = SP_3P_1,\\ SA_3P_2 &= SP_2A_1 = SP_1P_3,\end{aligned}$$

all of which, when A_1 and A_3 are fixed, are constant angles. Equations (9) furnish a proof for a well-known method of constructing the parabola.

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[*] The letter X, which through an oversight is wanting in the diagram, is any point of A_3P_2 extended.

Thus far the consideration has been entirely general, the points of contact being any points on the curve. The special cases are significant as indicative of the neatness and simplicity of quaternion methods; a few will therefore be given.

1. *When* ρ_2 *is a multiple of* ρ_1, *or when* $\rho_2 - \rho_1$ *is a focal chord,* we have $x\rho_2 = \rho_1$, in which the sign of x is essentially negative.

Then by (1)

$$x\left[\frac{1}{4a}(y_2^2 - 4a^2)\,\alpha + y_2\beta\right] = \frac{1}{4a}(y_1^2 - 4a^2)\,\alpha + y_1\beta\,,$$

$$\therefore\quad x = \frac{y_1^2 - 4a^2}{y_2^2 - 4a^2} = \frac{y_1}{y_2}\,,\quad y_1(y_1y_2 + 4a^2) = y_2(y_1y_2 + 4a^2)\,,$$

which can only be satisfied, under the given conditions, by $y_1y_2 + 4a^2 = 0$.

Then, by (1) and (3),

$$S\pi_3\rho_1 = -\frac{1}{16a^2}(y_1^2 + 4a^2)(y_1y_2 + 4a^2) = 0\,,$$

or π_3 is perpendicular to ρ_1; *i. e. the line from the focus to the intersection of the tangents at the extremities of a focal chord is perpendicular to the focal chord.*

Also, by (5),

$$S(\rho_1 - \pi_3)(\rho_2 - \pi_3) = S\rho_1\rho_2 + \pi_3^2 = 0\,,$$

or *the tangents at the extremities of a focal chord are perpendicular to each other.*

Since $y_1y_2 + 4a^2 = 0$, we have

$$\pi_3 = \frac{1}{4a}(y_1y_2 - 4a^2)\,\alpha + \tfrac{1}{2}(y_1 + y_2)\,\beta = -2a\alpha + \tfrac{1}{2}(y_1 + y_2)\,\beta\,,$$

i. e. the tangents at the extremities of a focal chord meet on the directrix.

2. *When* ρ_2 *becomes a multiple of* β,

$$\rho_2 = \frac{1}{4a}(y_2^2 - 4a^2)\,\alpha + y_2\beta = x\beta\,,$$

$$x = y_2 = \pm 2a\,;$$

i. e. the parameter equals the double ordinate through the focus, or twice the distance from the focus to the directrix.

3. *When* ρ_2 *becomes a multiple of* α, $y_2 = 0$, *and*

$$\pi_3 = -a\alpha + \tfrac{1}{2}y_1\beta\,;$$

i. e. the subtangent is bisected at the vertex.

Also, $\pi_3 - \rho_1 = -a\alpha + \frac{y_1}{2}\beta - \left(\frac{y_1^2 - 4a^2}{4a}\alpha + y_1\beta\right) = -\frac{1}{4a}y_1^2\,\alpha - \tfrac{1}{2}y_1\beta$,

$$S\pi_3(\pi_3 - \rho_1) = 0\,;$$

i. e. the perpendicular from the focus on a tangent meets it on the principal tangent.

Again, the normal at A_1 may be written

$$x\pi_3 = x\,(-a\alpha + \tfrac{1}{2}\,y_1\beta) = z\alpha + y_1\beta, \quad x = 2; \quad z = -2a;$$

showing that *the subnormal is constant;* and that *the normal is twice the perpendicular from the focus on the tangent.*

Also, $x\pi_3 = -z\alpha + \rho_1$, $x\,(-a\alpha + \tfrac{1}{2}\,y_1\beta) = -z\alpha + \dfrac{1}{4a}\,(y_1^2 - 4a^2)\,\alpha + y_1\beta$,

$$x = 2; \quad z = \frac{1}{4a}\,(y_1^2 + 4a^2) = \mathrm{T}\rho_1,$$

which shows that *the distance from the foot of the normal to the focus equals the radius vector to the point of contact, or the distance from the foot of the tangent to the focus, or the distance from the point of contact to the directrix;* proofs for each of which may also be obtained directly by writing the proper vector equalities as obtained from the figure and determining the value of the unknown coefficients as above.

The part of the tangent from its foot to the point of contact is readily found to be $\dfrac{1}{2a}\,y_1^2\alpha + y_1\beta$, since it is equal to $z\alpha + \rho_1$, and z has the value $\dfrac{1}{4a}\,(y_1^2 + 4a^2)$, as already shown in our last equation. The part of the tangent from the foot of the focal perpendicular to the point of contact is

$$-\pi_3 + \rho_1 = \frac{1}{4a}\,y_1^2\alpha + \tfrac{1}{2}\,y_1\beta,$$

i. e. the tangent is bisected by the focal perpendicular. Whence *the angle between the tangent and the radius vector equals the angle between the tangent and the axis, or the angle between the normal and the directrix;* and also, *the tangent bisects the angle between the diameter and the focal radius to the point of contact.*

The perpendicular from the focus on the normal is also $-\pi_3 + \rho_1$, whose expression just given proves that *the locus of the foot of the perpendicular from the focus on the normal is a parabola, whose vertex is at the focus of the given parabola, and whose parameter is $\tfrac{1}{4}$ of that of the given parabola.*

The vector to the middle point of a focal chord (when ρ_2 is a multiple of ρ_1) is

$$\rho = \tfrac{1}{2}\,(\rho_1 + \rho_2) = \tfrac{1}{2}\left(1 + \frac{y_2}{y_1}\right)\,\rho_1 = \frac{1}{2y_1^2}\,(y_1^2 - 4a^2)\,\rho_1$$

$$= \frac{1}{8ay_1^2}\,(y_1^2 - 4a^2)^2\,\alpha + \frac{1}{2y_1}\,(y_1^2 - 4a^2)\,\beta,$$

that is, *the locus of the middle points of focal chords is a parabola, whose vertex is at the focus of the given parabola, and whose parameter is the parameter of the given parabola.*

4. *When* π_3 *becomes a multiple of* β;

$$\pi_3 = \frac{1}{4a}(y_1y_2 - 4a^2)\,\alpha + \tfrac{1}{2}(y_1 + y_2)\,\beta = x\beta$$

$$\therefore\ y_1y_2 = 4a^2,\ x = \tfrac{1}{2}(y_1 + y_2) = \frac{1}{2y_1}(y_1^2 + 4a^2)\,,$$

$$\pi_3 = \frac{1}{2y_1}(y_1^2 + 4a^2)\,\beta\,,\ \ \pi_3^2 = -\frac{1}{4y_1^2}(y_1^2 + 4a^2)^2.$$

When the point P_3 *coincides with the intersection of the tangent* A_1P_3 *with the directrix*, we have

$$\pi_3 = \frac{y_1y_2 - 4a^2}{4a}\,\alpha + \frac{y_1 + y_2}{2}\,\beta = -2a\alpha + x\beta\,,$$

$$\therefore\ y_1y_2 = -4a^2,\ x = \frac{y_1 + y_2}{2} = \frac{1}{2y_1}(y_1^2 - 4a^2)\,,$$

$$\pi_3 = -2a\alpha + \frac{1}{2y_1}(y_1^2 - 4a^2)\,\beta\,,\ \ \pi_3^2 = -\frac{1}{4y_1^2}(y_1^2 + 4a^2)^2,$$

hence, *any tangent intersects the latus rectum and directrix in points equally distant from the focus.*